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Asymptotic Geometry of Hyperbolic Well-Ordered Cantor Sets

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In this paper we study the well-ordered Cantor sets in hyperbolic sets on the line and the plane. Examples of such sets occur in circle maps and in area-preserving twist maps. We set up a renormalization scheme employing in both cases the first return map. We prove convergence of this scheme. The convergence implies that the asymptotic geometry of such a well-ordered set with irrational rotation number and their nearby well-ordered orbits is determined by the Lyapunov exponent of this set.

KEY WORDS: Renormalization, hyperbolic Cantor sets, Lyapunov exponents, bounded nonlinearity, Denjoy-Koksma, Aubry-Mather sets, asymptotic geometry.

0. INTRODUCTION

In this paper we study analytic aspects of well-ordered Cantor sets in one- and two-dimensional hyperbolic sets. The general problem is the following. Such well-ordered Cantor sets have a well-defined rotation number. Each such well-ordered Cantor set can be approximated by well-ordered periodic orbits. If one chooses the rotation number of these periodic orbits to approximate the rotation number of the given Cantor set very well, one then expects that the corresponding periodic set approximates the Cantor set very well. Moreover, in the hyperbolic setting this convergence should be controlled by the positive Lyapunov exponent $\lambda(E)$ of the Cantor set E.[15] In this paper we study a class of such hyperbolic sets, arising from maps for which well-orderedness can be defined. Now fix an irrational

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minimal well-ordered Cantor set in this hyperbolic set. We set up a renormalization scheme defined by the symbolic dynamics at special points in the Cantor set. The sequence of renormalized maps is constructed by considering first return maps, as it is done for one-dimensional maps. It turns out this sequence depends essentially only on the Lyapunov exponents of the well-ordered set. We show that the sequence of renormalizations converges at a superexponential rate to the sequence of renormalizations of a linear map (Theorems 2.8 and 3.8). For the definition of convergence of renormalizations see the Appendix. Here we assume that the maps under consideration are of class C^2. In the one-dimensional case $C^{1+\alpha}$ suffices. (We say that the asymptotic geometry of the original Cantor set is linear.) As a corollary of this method one obtains:

Theorem 2.4. (One-dimensional.) Let E_α be a well-ordered minimal Cantor set for a smooth $(C^{1+\beta})$ one-dimensional expanding map of rotation number α. Let p/q be a rational approximant of α. Let $E_{p/q}$ be the approximating well-ordered periodic orbit of rotation number p/q. Then

$$d_{\mathrm{H}}(E_{p/q}, E_\alpha)e^{q\lambda(\alpha)}$$

is uniformly in q bounded away from zero and infinity.

Here d_{H} denotes the Hausdorff distance on sets.

This paper was written as a sequel ro ref. 15. In that paper we study hyperbolic Aubry–Mather sets for area-preserving monotone twist maps. We show there how, under certain geometric assumptions, one can define a renormalization scheme for such hyperbolic Aubry–Mather sets. Using the results of ref. 16, we can prove that these assumptions are satisfied for the standard map with large nonlinearity parameter. The results of the present paper imply convergence of this renormalization scheme.

In the area-preserving case the stable and unstable Lyapunov exponents λ^s and λ^u of a minimal hyperbolic set are the same in absolute value. One then obtains the analogous statement to Theorem 2.4 concerning the speed of convergence of certain well-ordered periodic orbits to Aubry–Mather sets of irrational rotation number. More precisely:

Corollary. Let E_α be a hyperbolic Aubry–Mather set of rotation number α for the standard map (with large nonlinearity parameter). Let p/q be a rational approximant of α. Let $E_{p/q}$ be the approximating well-ordered periodic Aubry–Mather set of rotation number p/q. Then

$$d_{\mathrm{H}}(E_{p/q}, E_\alpha)\, e^{q\lambda^u(\alpha)/2}$$

is uniformly in q bounded away from zero and infinity.

The setup of this paper is as follows. In Section 1 we recall the construction of the symbolic dynamics for such well-ordered Cantor sets. In Section 2 we study the analysis in the one-dimensional setting. In Section 3 we study the two-dimensional case.

1. SYMBOLIC DYNAMICS OF EXPANDING WELL-ORDERED SETS

In this section we review for future purposes the symbolic dynamics of well-ordered Cantor sets. We introduce at the end of this section the topological format of a renormalization scheme.

Consider disjoint intervals I_0, $I_1 \subset I \subset \mathbf{R}$ and expanding, orientation-preserving homeomorphisms

$$f_i\colon \quad I_i \to f_i(I_i) = I, \qquad i = 0, 1$$

Define $f\colon I_0 \cup I_1 \to I$ as $f|_{I_i} = f_i$. We assume that f_0 fixes the left endpoint of I_0, and f_1 fixes the right endpoint of I_1. Assume that f is C^1 and $\|f'\| > \gamma > 1$. Then, as is well known, the nonwandering set $\Lambda(f)$ is a Cantor set.

Definition. An f-invariant set E in $\Lambda(f)$ is well ordered if $f|_E$ extends as a monotone circle map to I with the endpoints identified.

Each well-ordered set then has a well-defined rotation number in $\mathbf{R}/\mathbf{Z}$. The following proposition has been discovered by many people.(4, 10, 12, 13)

Proposition 1.1. If f preserves orientation, then for all $\alpha \neq 0$ in $\mathbf{R}/\mathbf{Z}$, f has a unique well-ordered minimal set E_α in $\Lambda(f)$ of rotation number α.

Such minimal sets are constructed as follows: Denote by

$$h\colon \quad \Sigma = \{0, 1\}^{\mathbf{N}} \to \Lambda(f)$$

the standard conjugacy between the shift map σ on Σ and f on $\Lambda(f)$:

$$h(s_0, s_1, \ldots, s_n, \ldots) = \bigcap_{i=0}^{\infty} f_{s_0}^{-1} \circ \cdots \circ f_{s_i}^{-1}(I)$$

Here f_i^{-1} denote the two right inverses of f.

Remark. Provided the context is clear, we name a subset in $\Lambda(f)$ by the corresponding set of sequences in Σ.

Provide Σ with the dictionary topology $(0 < 1)$. Provided that I_1 is to the right of I_0, h is order preserving, since f is orientation preserving.

For a real number x define $I(x)$ to be its integer value $= \max\{n \mid n \leqslant x\}$.

Fix $\alpha \neq 0$. For $0 \leqslant d \leqslant 1$, consider the line with equation $y = \alpha x + d$. To each such d we will associate a sequence of zeros and ones, which we denote by $s_\alpha(d)$. The ith symbol of this sequence is defined as follows:

$$s_\alpha(d)_i = I(\alpha(i+1)+d) - I(\alpha i + d)$$

In other words, zero or one, depending on whether the integer value changes (see Fig. 1). So we have a map s_α: $[0, 1]/_{0=1} \to \Sigma$. Define $s_{\alpha,<}$ by the analogous receipe where one changes the definition of integer value to $\max\{n \mid n < x\}$. For sake of completeness we summarize the main observations:

1. (Monotonicity.) For α fixed, s_α is monotone in d. For d fixed, $s_\alpha(d)$ is monotone in α.

2. $s_\alpha \circ R_\alpha = \sigma \circ s_\alpha$ (translate unity to the left). Here $R_\alpha(d) = d + \alpha$. One has the analogous conjugacy for $s_{\alpha,<}$.

3. The set of d for which the line $y = \alpha x + d$ contains a lattice point in $\mathbf{Z}^+ \times \mathbf{Z}^+$ makes up precisely the points of discontinuity and s_α is *right* continuous, $s_{\alpha,<}$ is *left* continuous.

4. Denote by E_α the closure of the image of s_α; E_α is a minimal set for σ.

5. The rotation number α of E_α is the average number of ones in a string for a point in E_α.

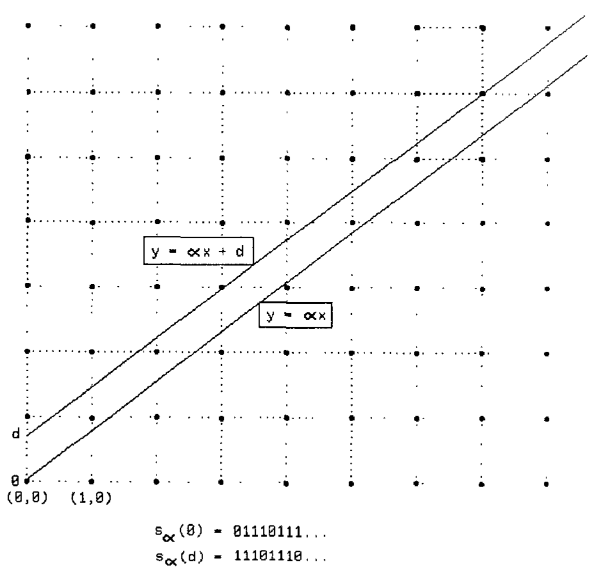

Fig. 1. The definition of $s_\alpha(d)$.

6. Define the endpoints of E_α to be those points which are not both right and left accumulation points. Since E_α is ordered, it makes sense to speak of gaps. For d a point of discontinuity, one has that $s_\alpha(d)$ denotes the left endpoint of the gap.

7. For α rational, E_α is a periodic orbit.

Fix α irrational, $\alpha = [a_0,..., a_n,...]$. We introduce the following notation. The sequence of continued fraction approximants to α is denoted by $\{p_n/q_n\}$. One has

$$p_{n+2} = a_{n+2} p_{n+1} + p_n, \qquad q_{n+2} = a_{n+2} q_{n+1} + q_n$$

The following proposition describes how well the orbit $s_{p_n/q_n}(0)$ approximates $s_\alpha(0)$.

Proposition 1.2. Let n be even, $p_n/q_n < \alpha$; then

$$\inf\{i \mid s_{p_n/q_n}(0)_i \neq s_\alpha(0)_i\} = q_{n+2} \geqslant 2q_n$$

Proof. This follows directly from the property of continued fractions as described, for example, in ref. 1. This can be proven as follows. We are considering two lines through the origin, one with slope α, the other with slope p_n/q_n. For continued fractions one has the following estimate on the denominators:

$$q_{n+2} \geqslant q_{n+1} + q_n \geqslant 2q_n$$

Since (q_{n+2}, p_{n+2}) is the first closest lattice point *below* the line slope α to the right of (q_n, p_n), the proposition follows. ∎

We want to describe the Cantor set E_α as an intersection of nested collections I_{q_n} of intervals in Σ. We have $E_\alpha = \bigcap_n I_{q_n}$. Here each I_q is a collection of intervals in Σ, determined by certain symbol sequences of length q, constructed as follows.

Definition. $I_K = \{s \in \Sigma \mid \text{first } K \text{ digits of } s \text{ equal first } K \text{ digits of } s_\alpha(d) \text{ for some } d\}$.

Consider the point $s_{p_n/q_n}(0)$. From the previous proposition it follows that this approximates the point $s_\alpha(0)$ very well. The extent to which its orbit approximates E_α is the content of the following lemma.

Lemma 1.3.

1. Every interval in $I_{q_n - 1}$ contains a single point in the orbit of $s_{p_n/q_n}(0)$.

2. All but one of the intervals in I_{q_n} contain a point in the orbit of $s_{p_n/q_n}(0)$.

Proof. See the Appendix in ref. 15. ∎

For later reference we want to understand how to construct the symbol sequence of s_{p_n/q_n}. Denote by T_n the segment of period q_n in its sequence.

Proposition 1.4. We have

$$\text{for } n \text{ even} \qquad T_{n+2} = T_n T_{n+1}^{a_{n+2}}$$

$$\text{for } n \text{ odd} \qquad T_{n+2} = T_{n+1}^{a_{n+2}} T_n$$

Proof. We will prove the first case. Consider the triangle with vertices $(0, 0)$, (q_n, p_n), and (q_{n+2}, p_{n+2}). This triangle contains no lattice point in its interior. The result then follows from the definition of $s_\alpha(0)$. ∎

Remark. These points are placed as follows:

$$\bullet_{2n} \quad \bullet_{2n+2} \quad \bullet_{\alpha} \quad \bullet_{2n+3} \quad \bullet_{2n+1}$$

We have denoted points by their subscripts.

We finish this section with a combinatorial version of renormalization in our setting. We will describe the construction of closest return maps on intervals bounded by periodic points. Pick two rational numbers $0 < p/q < r/s < 1$. Consider the periodic points P_0, respectively P_1, corresponding to $s_{p/q}(0)$ and $s_{r/s}(0)$. Their orbits are, by definition, well ordered. Consider the interval $\mathbf{J}$ in I bounded by these two periodic points. Define $\mathbf{J}_0$, respectively $\mathbf{J}_1$, to be the intervals $f^{-q}\mathbf{J} \cap \mathbf{J}$, respectively $f^{-s}\mathbf{J} \cap \mathbf{J}$. Now we can define new maps on $\mathbf{J}_0 \cup \mathbf{J}_1$ to $\mathbf{J}$ as f^q on $\mathbf{J}_0$ and f^s in $\mathbf{J}_1$. Denote this map by $R(f, \mathbf{J})$, the renormalization of f to the interval $\mathbf{J}$, and rescale $\mathbf{J}$ affinely to the unit interval. This renormalized map satisfies the same assumptions as our original map f. The map $R(f, \mathbf{J})$ is the (rescaled) first return map for those points in $\mathbf{J}$ which return in q or s iterates. For $R(f, \mathbf{J})$ we can define well-ordered sets, symbolic dynamics, etc.

Proposition 1.5. Assume $\det\begin{vmatrix} q & s \\ p & r \end{vmatrix} = 1$. Every minimal well-ordered set of $R(f, \mathbf{J})$ is in $\mathbf{J}$ contained in a minimal well-ordered set for f.

Proof. Let Δ_β be a well-ordered set for $R(f, \mathbf{J})$ in $\mathbf{J}$ of rotation number β with respect to $\mathbf{J}$. By iterating this set under f finitely many times, one obtains a minimal f-invariant set E.

We have to show that it is well ordered. The collection of symbolic sequences for E can be obtained as follows. Let $\underline{s}$ be a string for a point in Δ_β. Associate to $\underline{s}$ a new string $\underline{s}^*$ by substituting for each 0 in $\underline{s}$ the finite string for P_0, for each 1 the finite string for P_1. This defines a map * from the symbolic sequences of A_β into Σ. In terms of rotation numbers, the action of the map * is described by the following linear map A:

$$A\begin{bmatrix}0\\ -\\ 1\end{bmatrix}\to\begin{bmatrix}p\\ -\\ q\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}1\\ -\\ 1\end{bmatrix}\to\begin{bmatrix}r\\ -\\ s\end{bmatrix}$$

By assumption, this linear transformation has determinant one.

Denote $A[\beta/1]$ by β'. Then A maps the line $y=\beta x$ to the line $y=\beta' x$. Since A preserves orientation, lattice points above (below) the line $y=\beta x$ are mapped to lattice points above (below) $y=\beta' x$. Since this unimodular transformation A moreover maps the Farey tree into a subtree of itself, continued fraction approximants to β are mapped to continued fraction approximants to β'. Therefore $s_\beta(0)^*=s_{\beta'}(0)$. Since the set of symbolic sequences for E equals the closure of the union of all shifts of $s_\beta(0)^*$, we obtain that $E=E_{\beta'}$. ∎

This proposition implies that one can analyze well-ordered sets for f, using this renormalization construction, if one chooses approximating rationals suitably. For example, consecutive continued fraction approximants or consecutive Farey approximants satisfy the assumption of the proposition. In the next section we discuss analytic properties of these renormalizations.

2. ANALYSIS ON EXPANDING WELL-ORDERED SETS ON THE LINE

We assume that we are in the setting of the previous section: we are given two intervals I_i, $i=0,1$, on the real line and an orientation-preserving expanding map f defined on each of these intervals so that the image of each of these intervals contains both.

Remark. Let f and g be two such expanding maps. f and g are topologically conjugate on their nonwandering sets. We fix the topological conjugacy h by requiring it to be order preserving. Since f and g are both C^1, they have derivatives bounded away from 1 and ∞; it follows that h is already Hölder continuous. As a matter of fact, the modulus of continuity of h is at least

$$\min\left\{\frac{\min \ln f'_0}{\max \ln g'_0}, \frac{\min \ln f'_1}{\max \ln g'_1}\right\}$$

Here the subscripts denote the restriction of the map to their intervals I_0 and I_1.

From now on denote by E_α the unique well-ordered minimal set of rotation number α in $\Lambda(f)$, the nonwandering set of f. Now let ϕ be a Borel-measurable function on $\Lambda(f)$. Consider the function

$$I(\phi, \cdot): \qquad S^1 \to \mathbf{R} \cup \{\infty\}, \qquad I(\phi, \alpha) = \int \phi\mu_\alpha$$

Here μ_α denotes the unique f-invariant probability measure on E_α. This measure can be characterized as follows: Let ψ: $E_\alpha \to S^1$ be a semiconjugacy beween f on E_α and R_α on S^1. Then $\psi_* \mu_\alpha$ is Lebesgue measure. The collection of measures $\{\mu_\alpha\}$ is weak *-continuous at irrationals. Therefore the function $I(\phi, \cdot)$ is already continuous at irrationals for ϕ moderately regular. In the well-ordered case an important principle to obtain understanding of the behavior of the function $I(\phi, \cdot)$ is the Denjoy–Koksma theorem (see, for example, ref. 5). In the case where one has the additional information that the system is expanding, much stronger tools are available, for example, Renyi's discovery concerning bounded nonlinearity of compositions of $C^{1+\varepsilon}$ expanding maps with small image,[8] which we will use over and over again.

Denote by $C^\beta\Lambda(f)$ the Banach space of functions ϕ on $\Lambda(f)$, which are Hölder continuous of exponent β; denote by

$$|\phi|_\beta = \sup_{x,\, y \in \Lambda(f)} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^\beta}$$

the norm of ϕ. From now on the standing assumption is that f is C^1 and that $\inf f' \geqslant \gamma > 1$. The first proposition is concerned with how well finite time average converge to the actual average.

Proposition 2.1. (Hyperbolic Denjoy–Koksma.) Let α be irrational, p/q a rational approximant of α. Assume ϕ is in $C^\beta\Lambda(f)$. Let x_0 be a point in E_0; then

$$\left| \sum_{i=0}^{q-1} \phi(f^i(x_0)) - q \int \phi\mu_\alpha \right| \leqslant \frac{q}{\gamma^{q\beta}}\, |\phi|_\beta$$

In particular, the time average converges, not just μ_α almost everywhere.

Proof. By Lemma 1.3, we have that E_α is contained in I_{q-1}, which consists of q intervals $\{I^i_{q-1}\}$ and each of such intervals consists of points whose first itinerary of length q is the same. Consequently each of these intervals has length smaller than γ^{-q}.

Now pick any point x_0 in E_α. Its first q iterates $(x,..., f^{q-1}(x))$ land in each of these intervals. So $\mu_\alpha(I^i_{q-1}) = 1/q$.

Consider again these first $q-1$ iterates. After relabeling we have $f^i(x_0) \in I^i_{q-1}$. Then

$$\left|\sum_{i=0}^{q-1} \phi(f^i(x_0)) - q\int \phi\mu_\alpha\right| \leqslant \sum_{i=0}^{q-1} \left|\phi(f^i(x_0)) - q\int_{I^i_{q-1}} \phi\mu_\alpha\right|$$
$$\leqslant \sum_{i=0}^{q-1} q\left|\int_{I^i_{q-1}} \{\phi(f^i(x_0)) - \phi\}\,\mu_0\right|$$

Because we have the estimate for the length of I^i_{q-1}, the estimate follows.

That this implies that the time average converges can be seen as follows. Let $\alpha = [a_0, a_1, \ldots, a_j, \ldots]$, and $N < q_n$. Then $N = \sum_{j<n} b_j q_j$ with $b_j \leqslant a_j$. Now we have

$$\left|\frac{1}{N}\sum_{i=0}^{N-1} \phi(f^i(x_0)) - \int \phi\mu_\alpha\right| \leqslant \frac{\sum_{j<n} b_j q_j \gamma^{-q_j\beta}}{\sum_{j<n} b_j q_j} |\phi|_\beta$$

Now the right-hand side converges to zero. ∎

We now immediately have the following.

Proposition 2.2. Let α be irrational, and p/q a rational approximant of α. Then for ϕ in C^β we have

$$|I(\phi, p/q) - I(\phi, \alpha)| \leqslant \frac{2}{\gamma^{q\beta}} |\phi|_\beta$$

Proof. Assume $p/q < \alpha$; the other case is treated analogously. For any point Q in $E_{p/q}$ on has

$$I\left(\phi, \frac{p}{q}\right) = \frac{1}{q}\sum_{0}^{q-1} \phi(f^i(Q))$$

We take Q in $E_{p/q}$ to be $s_{p/q}(0)$ and $x_0 = s_\alpha(0)$ in E_α. From the symbolic dynamics one then obtains that the first q iterates of Q are very close to the first q iterates of x_0; more precisely (see Proposition 1.2)

$$\text{for} \quad i = 1, \ldots, q-1: \quad |f^i(x_0) - f^i(Q)| \leqslant \frac{1}{\gamma^q}$$

Consequently

$$\left|\sum_{0}^{q-1} \phi(f^i(x_0)) - \phi(f^i(Q))\right| \leqslant \frac{q}{\gamma^{q\beta}} |\phi|_\beta$$

Now apply the previous proposition. ∎

An important application for our purpose concerns Lyapunov exponents. More precisely, assume that $f \in C^{1+\beta}$; let $\phi = \ln f'$. Then $I(\phi, \alpha)$ equals the Lyapunov exponent of f on E_α, which we will denote by $\lambda(\alpha)$. We remark that in this case, we can apply Proposition 2.1 and obtain that $\lambda(\alpha)$ equals the Lyapunov exponent of every point in E_α.

Corollary 2.3. Assume that f is of class $C^{1+\beta}$. For p/q a rational approximant to α

$$|\lambda(p/q) - \lambda(\alpha)| \leqslant |f'|_\beta / \gamma^{q\beta}$$

Now denote by d_{H} the Hausdorff distance on compact sets.

Theorem 2.4. Assume $f \in C^{1+\beta}$. Let p/q be a continued fraction approximant of α; then $d_{\mathrm{H}}(E_{p/q}, E_\alpha)e^{q\lambda(\alpha)}$ is uniformly bounded away from zero and infinity.

Proof. We have from Lemma 1.3 that $E_\alpha \subset I_{q-1}$. Recall that I_{q-1} consists of q intervals each containing one point in the orbit of $Q = s_{p/q}(0)$. Denote by $|I|$ the length of an interval I.

We first show that $|I^i_{q-1}| e^{q\lambda(p/q)}$ is (for p/q a rational approximant) bounded away from zero and infinity, uniformly in q. This can be seen as follows. Since f^{q-1} is injective on I^i_{q-1} and $f \in C^{1+\beta}$, we have that for all points x, y in I^i_{q-1}

$$\begin{aligned}
&|\ln[f^{(q-1)\prime}(x)] - \ln f^{(q-1)\prime}(y)| \\
&\quad \leqslant \sum_{i=0}^{q-2} |\ln f'(f^i(x)) - \ln f'(f^i(y))| \\
&\quad \leqslant |\ln f'|_\beta \sum_i |f^i(x) - f^i(y)|^\beta \\
&\quad \leqslant |\ln f'|_\beta \, |f^{q-1}(x) - f^{q-1}(y)|^\beta \sum^{q-2} \gamma^{-\beta i}
\end{aligned}$$

The last inequality holds because $|f^{q-1}(x) - f^{q-1}(y)|$ is no biger than 1.

Let $y = f^i(Q)$. This point is periodic and $f^{q\prime}(y) = e^{q\lambda(p/q)}$. For x in I^i_{q-1} the ratio $f^{q\prime}(x)/e^{q\lambda(p/q)}$ is then uniformly (in q) bounded away from zero and infinity. Since $f^{q-1}(I^i_{q-1})$ has length of order 1 (independent of q), we obtain that $|I^i_{q-1}| e^{q\lambda(p/q)}$ is uniformly bounded away from zero and infinity. Therefore (by Lemma 1.3) $d_{\mathrm{H}}(E_{p/q}, E_\alpha)$ is bounded from above by $e^{-q\lambda(p/q)}$. According to the previous corollary, $e^{q\lambda(p/q)}/e^{q\lambda(\alpha)}$ is uniformly bounded. Consequently, $d_{\mathrm{H}}(E_{p/q}, E_\alpha)e^{q\lambda(\alpha)}$ is uniformly bounded away from infinity.

By Lemma 1.3, one interval in I_q does *not* contain a point in the orbit of Q. The length of this interval is $O(e^{-q\lambda(\alpha)})$. Therefore, $d_{\mathrm{H}}(E_{p/q}, E_\alpha)e^{q\lambda(\alpha)}$ is also uniformly bounded away from zero. ▮

Remarks. 1. If one considers Farey approximants instead of continued fraction approximants, the convergence is typically not this good (this of course only applies to irrationals of unbounded type).

2. The present discussion generalizes straightforwardly to the case of a finite number of intervals.

3. As a further application of the analytic theory, consider smoothy circle endomorphisms with critical points. There are many examples[2] where one can construct well-ordered minimal sets for such maps. Assume that such a set avoids a neighborhood of the critical set. By ref. 9, such sets are hyperbolic, and by ref. 14, these sets imbed as well-ordered minimal sets for smooth expanding circle maps. Consequently, the previous applies.

In the present context we want to describe analytic properties of the renormalization scheme outlined in Section 1. More specificaly, this scheme concentrates on points which are endpoints of gaps. So let us consider the point $Q_\infty =_{\mathrm{def}} s_\alpha(0)$ in E_α. Consider the sequence of continued fraction approximants p_n/q_n to α. Consider for n the point $Q_n = s_{p_n/q_n}(0)$. One has that Q_{2n} is to the left of Q_∞ and Q_{2n+1} is to the right of Q_∞. Now consider the interval $\mathbf{J}_n$ bounded by Q_{2n} and Q_{2n+1}. Define $\mathbf{J}_{\mathrm{n},0}$ as the interval $f^{-q_{2n}}\mathbf{J}_{\mathrm{n}}$. Now, $R(f, \mathbf{R}_{\mathrm{n}})$, the renormalization of f to $\mathbf{J}_{\mathrm{n}}$, satisfies the same assumptions as our original f. In particular, $R(f, \mathbf{J}_{\mathrm{n}})$ has again well-ordered minimal sets of a given rotation number. Every such minimal set defines by repeated application of the original map f a minimal set for f on the original interval. One observes that this induced minimal set is again well ordered (Proposition 1.5). As far as the rotation number is concerned, we have the following.[15]

Proposition 2.5. If E is a well-ordered set and has rotation number α for $R(f, \mathbf{J}_{\mathrm{n}})$, then the induced well-ordered set for f has rotation number

$$\frac{\alpha p_{2n+1} + (1-\alpha)\, p_{2n}}{\alpha q_{2n+1} + (1-\alpha) q_{2n}}$$

Proof. This follows immediately from the characterization of the rotation number as the average number of ones. ▮

Now define the nonlinearity of a map f as

$$N(f) = \sup_{x,y} \left| \frac{f'(x)}{f'(y)} - 1 \right|$$

(see also ref. 11). From the proof of Theorem 2.4 we have that the sequence of maps $R(f, \mathbf{J_n})$ has uniformly bounded nonlinearity (on each of the intervals on which it is defined).

We have the following stronger result.

Proposition 2.6. Let α be irrational, and $\{p_n/q_n\}$ be the sequence of continued fractions to α. The renormalizations $\{R(f, \mathbf{J_n})\}$ converge exponentially fast in n to the set of linear expanding maps on $\mathbf{J_n}$ with slopes $e^{q_{2n}\lambda(\alpha)}$ and $e^{q_{2n+1}}\lambda(\alpha)$.

Proof. We how first that the nonlinearity of the expanding maps $R(f, \mathbf{J_n})$ tends to zero as n tends to infinity. Consider two points x and y in, say, $\mathbf{J_{n,0}}$ (the other case being analogous). Then with $q = q_{2n}$, we have by the same argument as in Theorem 2.4 (the total nonlinearity of a composition is determined by the length of the *image*):

$$\begin{aligned}
&|\ln[f^{q\prime}(x)] - \ln[f^{q\prime}(y)]| \\
&\quad \leqslant |\ln f'|_\beta \sum^{q-1} \gamma^{-\beta_i} |f^{q-1}(x) - f^{q-1}(y)|^\beta \\
&\quad \leqslant |\ln f'|_\beta \sum^{q-1} \gamma^{-\beta i} |\mathbf{J_n}|^\beta
\end{aligned}$$

Recall that the length of the interval $\mathbf{J_n}$ tends to zero (exponentially fast) as n goes to infinity. This shows that for n large, $R(f, \mathbf{J_n})$ is approximately linear with slopes $f^{q_{2n}\prime}(Q_{2n})$ and $f^{q_{2n+1}\prime}(Q_{2n+1})$. Now apply the corollary ∎

Now consider the "linear" map $L(\alpha)$ in our class of maps defined as follows. $L(\alpha)$ is defined on two intervals in the unit interval I; it is linear on these intervals and the derivative is the same on these intervals, namely $\exp[\lambda(\alpha)]$; the point 0 and 1 are fixed. Now consider the subsequent renormalizations $R(L(\alpha), \mathbf{J}'_n)$, where $\mathbf{J}'_n$ are the corresponding intervals. Denote by H_n the topological conjugacy between the nonwandering sets of $R(f, \mathbf{J}_n)$ and $R(L(\alpha), \mathbf{J}'_n)$. For the terminology in the next theorem we refer to the Appendix.

Proposition 2.7:

1. The Hölder constant of H_n tends to 1 faster than $1 - \gamma^{-q_{2n}}$.

2. The Lipschitz distance between $\mathbf{J}_{n+1}$ in $\mathbf{J}_n$ and $\mathbf{J}'_{n+1}$ in $\mathbf{J}'_n$ goes to zero faster than $\gamma^{-q_{2n}}$.

Proof. This follows readily from the analysis as set up so far. ∎

Theorem 2.8. The sequence of renormalizations $\{R(f, \mathbf{J}_n)\}_0^\infty$ converges to the sequence of renormalizations $\{R(L(\alpha), \mathbf{J}_n)\}_0^\infty$ as n tends to infinity.

Remark. In principle, the speed of convergence can be estimated better. However, since the numerators in the continued fraction approximants to α already grow very fast, the difference will be hard to observe numerically. One notices, though, that even in the hyperbolic setting there is, as far as speed of convergence is concerned, still a noticeable difference between irrational numbers of bounded type and, say, Liouville numbers. For the latter the convergence is extremely fast.

This theorem implies that when one renormalizes at a gap point of the well-ordered Cantor set E_α, the geometry of E_α at this point is completely controlled by the Lyapunov exponent of this set. This asymptotic geometry is independent of the particular choice of gap point. The convergence is, however, not uniform.[15]

3. THE TWO-DIMENSIONAL CASE

In this section we will partially generalize the previous results to a class of two-dimensional hyperbolic sets, for which one can define a notion of well-orderedness. For a given well-ordered minimal set E_α, we will define a renormalization procedure analogous to the one-dimensional case. The renormalized maps will be defined on certain "rectangles" bounded by the local stable and unstable manifolds of two periodic points[15] (both of which are vertices of this "rectangle"). The sequence of rectangles determined by subsequent renormalizations is canonically determined by the "number theory" of α. The main result of this section is Theorem 3.12. It implies that the geometry of this sequence of rectangles (up to a global affine transformation) is determined exponentially fast by the "number theory" of α and the Lyapunov exponents of E_α. In particular, as far as this sequence of rectangles is concerned, its geometry is asymptotically converging to the geometry of the corresponding sequence of renormalizations in the case where the hyperbolic set is linear. Moreover, this theorem implies that subsequent renormalizations converge to the corresponding sequence of renormalizations one obtains in the linear case. That is to say: the Hölder exponent of corresponding conjugacies tends to 1 extremely fast.

Remark. Many of the results obtained in this section hold in a more general context. The way in which particular use has been made of the assumption that this renormalization process is concerned with well-ordered minimal sets is in the following. First of all, we have a version of (hyperbolic) Denjoy–Koksma for our setting. This basically amounts to saying that we know the invariant probability measure μ_α well enough to make a fairly precise statement concerning the existence and convergence of time averages. Moreover, in one part of the construction (Proposition 3.10) we use the projection maps obtained by pushing along the invariant foliations. Such foliations are typically not much better than C^1, and neither are such projections. In order to maintain bounds on the nonlinearity, it is therefore important not to have to use such projections very often.

We will now define a class of hyperbolic sets we want to consider. Consider rectangles I_i, $i \in \{0, 1\}$, in the square I (see Fig. 2). Assume we are given maps $f_i: I_i \to f_i(I_i) \subset I$ as indicated: both maps have a fixed point, both are orientation-preserving diffeomorphisms (C^2), and f_0 maps the interval I_0 all the way across I along the bottom, and f_1 maps the rectangle I_1 all the way across I along the top. We moreover assume that these maps are C^2 and uniformly hyperbolic: there exist smooth cone fields C^u and C^s on I which are strictly mapped into themselves by Df, resp. Df^{-1}: For $x \in I_0 \cup I_1$, Df_x strictly maps $C^u(x)$ into $C^u(f(x))$, for x in $f(I_0 \cup I_1)$, Df_x^{-1} maps $C^s(x)$ strictly into $C^s(f^{-1}(x))$. Here strictly means in terms of a fixed norm $|\cdot|$ on tangent vectors: if $v \in C^u(x)$, then $|Df_x v| \geqslant \gamma |v|$ for some $\gamma > 1$.

From these assumptions one obtains that the nonwandering set $\Lambda(f)$ is a hyperbolic Cantor set. In this setting one has on $\Lambda(f)$ stable and unstable bundles E^s and E^u and one has local stable and unstable manifolds tangent to these distributions. Since we are in the two-dimen-

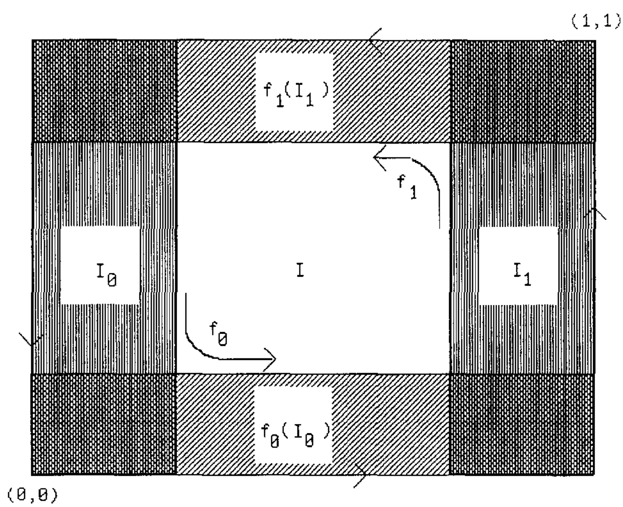

Fig. 2. The geometric definition of the maps f_0 and f_1.

sional case, these bundles are C^1.[6] Moreover, under the present geometric assumptions we have that these bundles have f-invariant orientations.

In the present setting one can once again define well-orderedness. Let V be a curve, say a stable manifold transverse to the unstable foliation, intersecting each leaf once. Denote by π^u the projection of $\Lambda(f)$ on V along the unstable foliation (π^s denotes the analogous projection along the stable direction). We say that a subset E of $\Lambda(f)$ is well ordered if the induced dynamics on the image of E under π^u imbeds in a monotone circle map. Using symbolic dynamics, it is again easy to trace well-ordered minimal sets of given rotation number. One can adopt the strategy of Section 1: fix a rotation number α and now one defines functions s_α, resp. $s_{\alpha,<}$, from S^1 to $\{0, 1\}^{\mathbf{Z}}$ (note the difference from dimension 1). All of the results of Section 1 carry over without any difficulty.

Proposition 3.1. For all $\alpha \neq 0$ the nonwandering set $\Lambda(f)$ contains a unique minimal well-ordered set E_α of rotation number α.

We similarly have the analog of hyperbolic Denjoy–Koksma:

Proposition 3.2. (Hyperbolic Denjoy–Koksma.) Let α be irrational, and p/q a rational approximant of α. Assume ϕ is in $C^\beta \Lambda(f)$. Let x_0 be a point in E_α; then

$$\left| \sum_{i=0}^{q-1} \phi(f^i(x_0)) - q \int \phi \mu_\alpha \right| \leqslant \frac{q}{\gamma^{q\beta/2}} |\phi|_\beta$$

In particular, the time average converges (not just μ_α almost everywhere).

Proof. We again want to find sets of small diameter and of μ_α measure $1/q$. In order to obtain sets of small diameter, we have to take into account forward and backward iterates. The construction of such sets is as follows. Let Q be a point in $E_{p/q}$ (here we assume that q is even; the odd case is treated similarly). Consider the set of points in $\Lambda(f)$ whose symbol sequences agree for $i = -q/2$ to $i = +q/2$ with the symbol sequence for Q. Then the diameter of this set is bounded by $\gamma^{-q/2}$. Moreover, by the analog of Lemma 1.3, we obtain that the μ_α measure of this set is $1/q$. Now we can repeat the proof of Proposition 2.1. ∎

Remark. Proposition 2.2 carries over without any difficulty (replace γ by $\gamma^{1/2}$).

Denote by $\mathbf{U}, \mathbf{S}$ the partial unstable, respectively stable, foliation on $I_0 \cup I_1$, defined as $W^u(\Lambda(f)) \cap I_0 \cup I_1$, respectively $W^s(\Lambda(f)) \cap I_0 \cup I_1$. We use the word "partial" since they are only defined on a subset. Denote also by $f^{-1}\mathbf{U}$, respectively $f\mathbf{S}$, the images of these foliations in I under f^{-1}, respectively f. Denote by f_u, respectively f_s, the induced C^2 maps on the leaves, $f_u = f$: $f^{-1}\mathbf{U} \to \mathbf{U}$ and $f_s = f^{-1}$: $f\mathbf{S} \to \mathbf{S}$, considered as one-dimensional maps. Parametrize all the leaves in each of the partial foliations by arc length.

Now let E_α be a well-ordered minimal set. Denote by $\lambda_u(\alpha)$, resp. $\lambda_s(\alpha)$, its unstable, resp. stable, Lyapunov exponent. Then $\lambda_u(\alpha) = \int \ln f'_u \mu_\alpha$ and $\lambda_s(\alpha) = -\int \ln f'_s \mu_\alpha$. Using Proposition 3.2 and the analog of Proposition 2.2, we have that for p/q a rational approximant $|\lambda_u(\alpha) - \lambda_u(p/q)|$ and $|\lambda_s(\alpha) - \lambda_s(p/q)|$ are exponentially small in q.

We want to define the nonlinearity of f_u, resp. f_s. Define the nonlinearity of f_u as the supremum over all connected leaves in $f^{-1}\mathbf{U}$ of the one-dimensional nonlinearity per leaf. Analogously for f_s.

Proposition 3.3. f_u and f_s have bounded nonlinearity.

Proof. This follows from the fact that the curvature of the local leaves is bounded. ∎

As in the one-dimensional case, it is important to be able to control the nonlinearity after many iterates of f. Let $\mathbf{U}_1$ and $\mathbf{U}_2$ be partial foliations both contained in $\mathbf{U}$. Assume that f_u^q: $\mathbf{U}_1 \to \mathbf{U}_2$ is well defined, i.e., f_u^q maps leaves in the first partial foliation into leaves of the second partial foliation and both foliations are local. Define $\mathbf{S}_1$ and $\mathbf{S}_2$ analogously.

Proposition 3.4. The nonlinearity of f_u^q (f_s^q) is bounded by a constant times the diameter of $\mathbf{U}_2$ ($\mathbf{S}_2$).

Proof. Here the diameter of a one-dimensional foliation is by definition the length of its longest leaf.

It is sufficient to prove the result for f_u. Now, f_u is leafwise C^2 and we can repeat the first part of the proof of Proposition 2.6. ∎

We finally need to discuss projection (holonomy) maps obtained by pushing along stable or unstable foliations (see Fig. 3). Let L be a leaf of $\mathbf{U}$. Let V_1 and V_2 be smooth curves intersecting L transversely. Near $V_1 \cap L$ one can consider the projection from $V_1 \cap \mathbf{U}$ to $V_2 \cap \mathbf{U}$ defined by pushing along the leaves of $\mathbf{U}$. Since the partial foliation $\mathbf{U}$ is C^1, this holonomy map is C^1. In particular, if V_1 and V_2 are C^1-close, this map will have derivative close to one (again the size of the derivative is measured in terms of arc-length coordinates). The same discussion holds for pushing along the stable foliation. Such projections, which are initially only defined on a Cantor set, have C^1 extensions of derivative close to the derivative on the Cantor set.

In this setting we can define a return map to a rectangle as

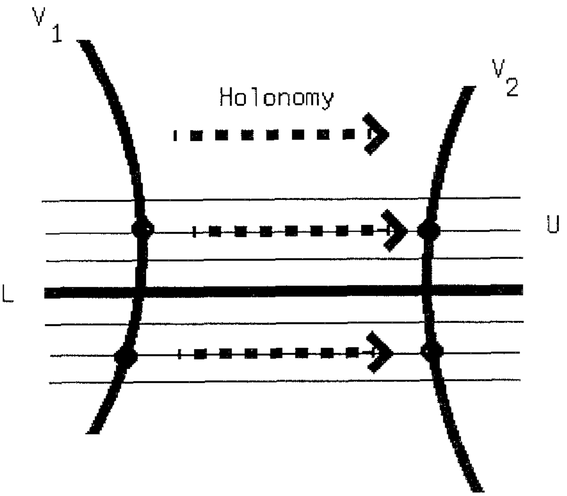

Fig. 3. The holonomy map: pushing along unstable leafs.

follows. From now on, we will again concentrate on continued fraction approximants p_n/q_n to α. We will renormalize on the points $Q_{2n} =_{\text{def}} s_{p_{2n}/q_{2n}}(0)$ and $Q_{2n+1} =_{\text{def}} s_{p_{2n+1}/q_{2n+1}}(0)$. Define the rectangle $\mathbf{J}_n$ as the diamond-shaped region whose boundaries are the local stable and unstable manifolds of these two points (note that Q_{2n} is the vertex at the lower left and Q_{2n+1} is at the upper right).

In this rectangle we define two strips $\mathbf{J}_{n,0}$ and $\mathbf{J}_{n,1}$: $\mathbf{J}_{n,0} =_{\text{def}} f^{-q_{2n}}(\mathbf{J}_n) \cap \mathbf{J}_n$ and $\mathbf{J}_{n,1} =_{\text{def}} f^{-q_{2n+1}}(\mathbf{J}_n) \cap \mathbf{J}_n$. Now define $R(f, \mathbf{J}_n)$ as the rescaled version of $f_{q_{2n}}$ on $\mathbf{J}_{n,0}$ and $f^{q_{2n+1}}$ on $\mathbf{J}_{n,1}$. The $R(f, \mathbf{J}_n)$ satisfies the assumptions of the map at the beginning of this section (see also Fig. 4). We call $R(f, \mathbf{J}_n)$ the renormalization of f on $\mathbf{J}_n$: it can be considered as the return map to $\mathbf{J}_n$. In particular, $R(f, \mathbf{J}_n)$ will have well-ordered minimal sets. We remark that for general $p/q < r/s$, such rectangles and renormalizations can be defined analogously.

Concerning the shape of $\mathbf{J}_n$ we want to make a few remarks. One observes that the symbol sequences for Q_{2n} and Q_{2n+1} agree for $i = -q_{2n} + 1$ to $i = q_{2n-1} - 1$. This implies that $\mathbf{J}_n$ is a small and very skinny parallellogram, with angles determined by the intersection of local stable and unstable manifolds at the chosen point in E_α. The strips $\mathbf{J}_{n,0}$ and $\mathbf{J}_{n,1}$ are extremely skinny compared to $\mathbf{J}_n$.

Now consider the "linear" map $L(\alpha)$ in our class of maps defined as follows. $L(\alpha)$ is defined on two strips in the unit square I; it is linear on these strips and the derivative diagonal and the same on both of these strips: namely $\exp[\lambda_u(\alpha)]$ in the horizontal direction and $\exp[\lambda_s(\alpha)]$ in the

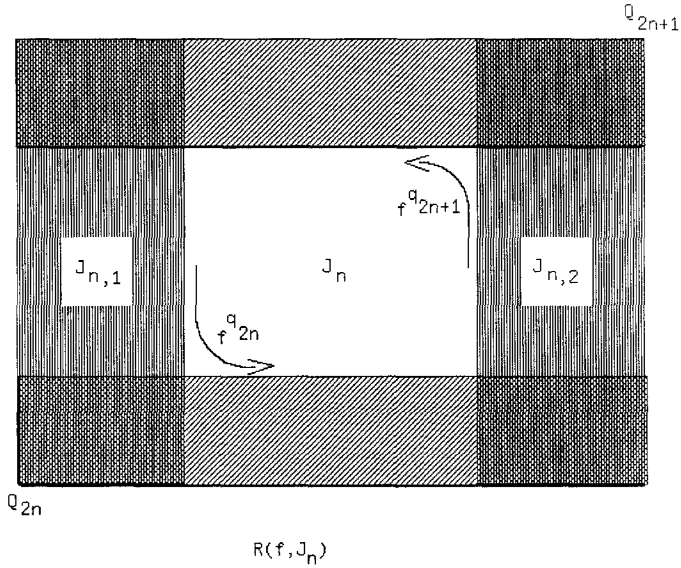

Fig. 4. The geometric definitions of the nth renormalization $R(f, \mathbf{J}_n)$.

vertical direction. The points (0, 0) and (1, 1) are fixed. Now consider the subsequent renormalizations $R(L(\alpha), \mathbf{J}_n')$, where $\mathbf{J}_n'$ are the corresponding rectangles. Denote by H_n the topological conjugacy between the non-wandering sets of $R(f, \mathbf{J}_n)$ and $R(L(\alpha), \mathbf{J}_n')$.

We first study the following one-dimensional problem (see Fig. 5).

Consider the map h_n, the restriction of H_n to $\mathbf{S}_n \cap W^u_{\mathrm{loc}}(Q_{2n})$, the "bottom" of $\mathbf{J}_n$. The map h_n conjugates $(f^u)^{q_{2n}}$ on I_0 to its linear equivalent on I_0' and $\pi^s \circ (f^u)^{q_{2n+1}} \circ \pi^{s-1}$ on the right interval I_1 to its linear equivalent on I_1'. Here π^s denotes the projection along the stable leafs in from "top" to "bottom" in $\mathbf{J}_n$.

Lemma 3.5. The conjugacy h_n is Hölder and its Hölder exponent is at least $1 - o(\gamma^{-q_{2n}})$ as n tends to infinity.

Proof. Since the rectangle $\mathbf{J}_n$ is exponentially small in n, we have that π^s has derivative very close to one.

Now h_n conjugates two one-dimensional maps of the type discussed in Section 2. Combining this with Proposition 3.4, we conclude that each of these one-dimensional maps has exponentially small nonlinearity. By the analog to Corollary 2.3, we have that the derivatives on corresponding intervals are exponentially close. Therefore, ratios of derivatives are very close to one (see the remark in the beginning of Section 2). ∎

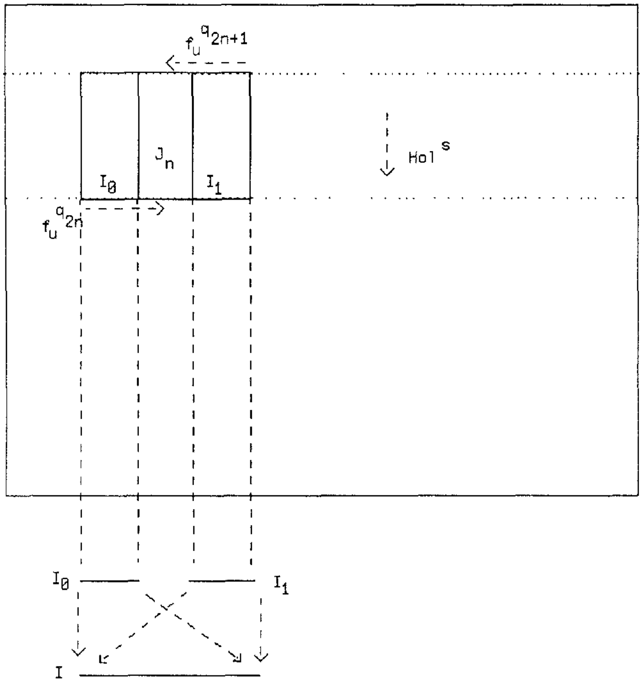

Fig. 5. Reduction to one-dimensional expanding maps.

Corollary 3.6. The Hölder constant of the conjugacy between $R(f, \mathbf{J}_n)$ and $R(L(\alpha), \mathbf{J}'_n)$ is at least $1 - o(\gamma^{-q_{2n}})$ as n tends to infinity.

Proof. To obtain H_n, push points along stable and unstable leaves and use the differentiability of the projection π^s. ∎

Proposition 3.7. The Lipschitz distance between $\mathbf{J}_{n+1}$ in $\mathbf{J}_n$ and $\mathbf{J}'_{n+1}$ in $\mathbf{J}'_n$ goes to zero faster than $\gamma^{-q_{2n}}$.

Proof. Consider the rectangle $\mathbf{J}_n$. In order to construct $\mathbf{J}_{n+1}$ it suffices to determine the leaves in $\mathbf{U}$ and $\mathbf{S}$ bounding it. See Fig. 6. Each of these leaves corresponds, as in the proof of Lemma 3.5, to fixed points of one-dimensional maps of bounded nonlinearity. ∎

We now reformulate the previous propositions in our theorem.

Theorem 3.8. The sequence of renormalizations $\{R(f, \mathbf{J}_n)\}_0^\infty$ converges to the sequence of renormalizations $\{R(L(\alpha), \mathbf{J}'_n)\}_0^\infty$ as n tends to infinity.

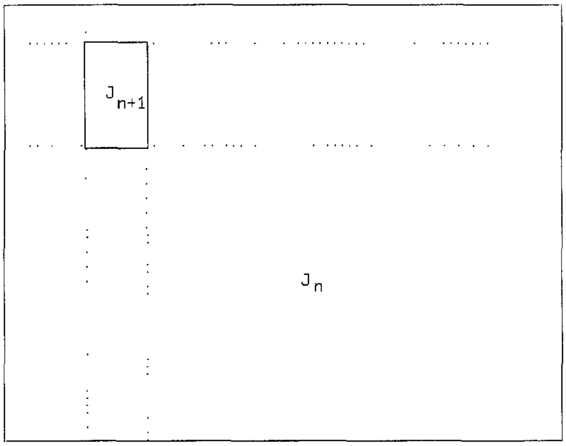

Fig. 6. The location of the $n+1$st domain in the nth domain.

Remark. As long as α is irrational, the sequence of renormalizations $\{R(f, \mathbf{J}_n)\}$ is well defined. The speed of the convergence is slowest, but still superexponentially convergent in n, for rotation numbers of bounded type.

APPENDIX. CONVERGENCE OF RENORMALIZATION

In this Appendix we present a definition of convergence of renormalizations appropriate for our context.

Each map f in the class of maps we consider in Sections 2 and 3 defines a sequence of renormalizations $\{R(f, \mathbf{J}_n)\}_0^\infty$. The domains $\mathbf{J}_n$ of definition for the renormalized maps depend on the initial choice of map and form a decreasing sequence of sets $\mathbf{J}_{n+1} \subset \mathbf{J}_n$.

In the one-dimensional case each of these intervals is bounded by two specific periodic points. In the two-dimensional setting (Section 3) each of these domains is a "rectangle" bounded by local unstable and stable manifolds of two specific periodic points (both of which are vertices of the rectangle). Moreover, the next domain $\mathbf{J}_{n+1}$ is in a very specific region of this "rectangle." To each of these rectangles $\mathbf{J}_n$, we can associate an affine transformation $A_{\mathbf{J}_n}$. This transformation $A_{\mathbf{J}_n}$ is determined by the following requirements: orientation preserving, the vertex Q_{2n} goes to $(0, 0)$, and the two adjacent points go to $(1, 0)$ and $(0, 1)$. The image of $\mathbf{J}_n$ under $A_{\mathbf{J}_n}$ converges exponentially fast in n to the unit square I. In the one-dimensional case this transformation $A_{\mathbf{J}_n}$ is determined by requiring it to be orientation preserving.

Consider for f and f' the sequences $\{R(f, \mathbf{J}_n)\}_0^\infty$ and $\{R(f', \mathbf{J}'_n)\}_0^\infty$.

In this setting we have the following: for each n, $R(f, \mathbf{J}_n)$ and $R(f, \mathbf{J}'_n)$ are, by assumption, topologically conjugate on their nonwandering set, by a transformation H_n. We have, moreover, that with respect to the Euclidean metric each H_n is Hölder continuous on the corresponding nonwandering set.

Definition (Convergence of renormalization). The sequence $\{R(f, \mathbf{J}_n)\}_0^\infty$ converges to the sequence $\{R(g, \mathbf{J}'_n)\}_0^\infty$ if:

1. The Hölder exponent of the conjugacy H_n converges to one as n tends to infinity.
2. The Lipschitz distance between $\mathbf{J}_{n+1}$ in $\mathbf{J}_n$ and $\mathbf{J}'_{n+1}$ in $\mathbf{J}'_n$ tends to zero as n tends to infinity.

The definition of (relative) Lipschitz distance[3] we use is the following.

Definition. Let M be a metric space with boundary $\partial(M)$, and A and B two homeomorphic subsets of M. Define the Lipschitz distance between A and B in M as

$$\inf\{\ln L(\varphi) + \ln L(\varphi^{-1}) \mid \varphi\colon (M, A) \to (M, B) \text{ is a homeomorphism, } \varphi = \mathrm{id} \text{ on } \partial(M)\}$$

Here $L(\varphi)$ denotes the infimum of the Lipschitz constants for φ. (If A and B are not Lipschitz homeomorphic, one defines their Lipschitz distance as $+\infty$.)

Now define the Lipschitz distance between J_{n+1} in J_n and J'_{n+1} in J'_n as the Lipschitz distance between $A_{\mathbf{J}_n}(\mathbf{J}_{n+1})$ and $A_{\mathbf{J}'_n}(\mathbf{J}'_{n+1})$ in the unit square I. (Note that the image of $\mathbf{J}_n$ itself under $A_{\mathbf{J}_n}$ converges exponentially fast in n to the unit square I.)

The point of this definition is that the sets $\mathbf{J}_{n+1}$, respectively $\mathbf{J}'_{n+1}$ have very small diameter with respect to $\mathbf{J}_n$, $\mathbf{J}'_n$. Moreover, $\mathbf{J}_{n+1}$ is also extremely close to the boundary of $\mathbf{J}_n$. If the Lipschitz distance between $\mathbf{J}_{n+1}$ in $\mathbf{J}_n$ and $\mathbf{J}'_{n+1}$ in $\mathbf{J}'_n$ is small, then in particular their locations in the respective bigger rectangles are comparable.

Remark. Part two of the definition of convergence of renormalization is a condition quite independent of part one. Although each conjugacy H_n is Hölder continuous of exponent close to one, this does not imply that the Lipschitz distance between $\mathbf{J}_{n+1}$ in $\mathbf{J}_n$ and $\mathbf{J}'_{n+1}$ in $\mathbf{J}'_n$ is small.

4. CONCLUDING REMARKS

In this paper we consider two examples of renormalization at points in well-ordered sets in certain hyperbolic maps. The main ingredient, besides hyperbolicity, is the hyperbolic Denjoy–Koksma theorem. The central reasons why we have such a theorem are that the invariant probability measure of a well-ordered set is concentrated on exponentially small intervals and the symbolic dynamics is very regular. It therefore seems reasonable to expect that a similar program can be carried out in different contexts. As an example, we mention the period doubling Cantor set in unimodal maps of positive entropy. This set is hyperbolic and has a fairly simple symbolic dynamics.

The fundamental problems in proving higher-dimensional generalizations (say, four-dimensional symplectic maps) are a lack in our understanding of the analogues of well-ordered behavior and the lack of smoothness of foliations.

Finally, we want to put the results of the renormalization approach described here in the context of renormalization of circle maps and twist maps. There one has with regard to renormalization of dynamics on well-ordered sets the following crude geometric picture in the space of such maps (unproven). There is a basin of attraction consisting of maps whose dynamics on a (given) well-ordered set is smoothly conjugate to a rigid rotation. Successive renormalizations of such a map converge to a set of maps whose dynamics is a rigid rotation on the corresponding well-ordered sets. Then there is a "codimension one" invariant set consisting of well-ordered sets on which the map is smoothly conjugate to a "critical" circle map. This critical set is normally repelling for the renormalization operator and forms the boundary of the basin of attraction described before. Its other side consists of maps whose (given) well-ordered set is a hyperbolic Cantor set. This is the side discussed here and one has that successive renormalizations in this region go off to infinity (scaling go at a superexponential rate to zero).

In this set our results show that for two maps with hyperbolic well-ordered sets of the same rotation number and the same Lyapunov exponent, successive renormalizations of the one converge to successive renormalizations of the other. [In the case where the rotation number is the golden mean and one can speak of fixed points of renormalization our results amount to studying the unstable manifold of the "critical" map in the neighborhood of infinity (R. MacKay, private communication).] The novel feature in this case is that renormalizations of two maps with well-ordered sets of the same rotation number, but different Lyapunov exponents, diverge. This second parameter (Lyapunov exponent) does not seem to have an analog in the other cases.

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